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# **Nashpy Documentation**

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This is a Python library used for the computation of equilibria in 2 player strategic form games.

This is a library with simple dependencies (it only requires `numpy` and `scipy`) so that it is pip installable.



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## Tutorial: building and finding the equilibrium for a simple game

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### 1.1 Introduction to game theory

Game theory is the study of strategic interactions between rational agents. Simply put that means that it's the study of interactions when the involved parties try and do what is best from their point of view.

As an example let us consider **Rock Paper Scissors**. This is a common game where two players choose one of 3 options (in game theory we call these *strategies*):

- Rock
- Paper
- Scissors

The winner is decided according to the following:

- Rock crushes scissors
- Paper covers Rock
- Scissors cuts paper

We can represent this mathematically using a 3 by 3 matrix:

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

The matrix  $A_{ij}$  shows the utility to the player controlling the rows when they play the  $i$  th row and their opponent (the column player) plays the  $j$  th column. For example, if the row player played Scissors (the 3rd strategy) and the column player played Paper (the 2nd strategy) then the row player gets:  $A_{32} = 1$  because Scissors cuts Paper.

A recommend text book on Game Theory is [Maschler2013].

## 1.2 Installing Nashpy

We are going to study this game using Nashpy, first though we need to install it. Nashpy requires the following things to be on your computer:

- Python 3.5 or greater;
- Scipy 0.19.0 or greater;
- Numpy 1.12.1 or greater.

Assuming you have those installed, to install Nashpy:

- On Mac OSX or linux open a terminal;
- On Windows open the Command prompt or similar

and type:

```
$ pip install nashpy
```

If this does not work, you might not have Python or one of the other dependencies. You might also have problems due to `pip` not being recognised. To overcome these, using the [Anaconda](#) distribution of Python is recommended as it installs straightforwardly on all operating systems and also includes the libraries needed to run Nashpy.

## 1.3 Creating a game

We can create this game using Nashpy:

```
>>> import nashpy as nash
>>> import numpy as np
>>> A = np.array([[0, -1, 1], [1, 0, -1], [-1, 1, 0]])
>>> rps = nash.Game(A)
>>> rps
Zero sum game with payoff matrices:

Row player:
[[ 0 -1  1]
 [ 1  0 -1]
 [-1  1  0]]

Column player:
[[ 0  1 -1]
 [-1  0  1]
 [ 1 -1  0]]
```

The string representation of the game also contains some information. For example, it is also showing the matrix that corresponds to the utility of the column player. In this case that is just  $-A$  but that does not always have to be the case.

We can in fact pass a pair of matrices to the game class to create the same game:

```
>>> B = - A
>>> rps = nash.Game(A, B)
>>> rps
Zero sum game with payoff matrices:

Row player:
```

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```
[[ 0 -1  1]
 [ 1  0 -1]
 [-1  1  0]]
```

Column player:

```
[[ 0  1 -1]
 [-1  0  1]
 [ 1 -1  0]]
```

We get the exact same game, if passed a single game, `Nashpy` will assume that the game is a *zero sum game*: in other words the utilities of both players are opposite.

## 1.4 Calculating the utility of a pair of strategies

If the row player played Scissors (the 3rd strategy) and the column player played Paper (the 2nd strategy) then the row player gets:  $A_{32} = 1$  because Scissors cuts Paper.

A mathematical approach to representing a strategy is to consider a vector of the size: the number of strategies. For example  $\sigma_r = (0, 0, 1)$  is the row strategy where the row player always plays their third strategy. Similarly  $\sigma_c = (0, 1, 0)$  is the strategy for the column player where they always play their second strategy.

When we represent strategies like this we can get the utility to the row player using the following linear algebraic expression:

$$\sigma_r A \sigma_c^T$$

Similarly, if  $B$  is the utility to the column player their utility is given by:

$$\sigma_r B \sigma_c^T$$

We can use `Nashpy` to find these utilities:

```
>>> sigma_r = [0, 0, 1]
>>> sigma_c = [0, 1, 0]
>>> rps[sigma_r, sigma_c]
array([ 1, -1])
```

Players can of course choose to play randomly, in which case the utility corresponds to the long term average. This is where our representation of strategies and utility calculations becomes particularly useful. For example, let us assume the column player decides to play Rock and Paper “randomly”. This corresponds to  $\sigma_c = (1/2, 1/2, 0)$ :

```
>>> sigma_c = [1 / 2, 1 / 2, 0]
>>> rps[sigma_r, sigma_c]
array([ 0.,  0.])
```

The row player might then decide to change their strategy and “randomly” play Paper and Scissors:

```
>>> sigma_r = [0, 1 / 2, 1 / 2]
>>> rps[sigma_r, sigma_c]
array([ 0.25, -0.25])
```

The column player would then probably deviate once more. Whether or not there is a pair of strategies for both players at which they both no longer have a reason to move is going to be answered in the next section.

## 1.5 Computing Nash equilibria

Nash equilibria is (in two player games) a pair of strategies at which both players do not have an incentive to deviate. We can find these using Nashpy:

```
>>> eqs = rps.support_enumeration()
>>> list(eqs)
[(array([ 0.333...,  0.333...,  0.333...]), array([ 0.333...,  0.333...,  0.333...]))]
```

*Nash* equilibria is an important concept as it allows to gain an initial understanding of emergent behaviour in complex systems.

How to:

## 2.1 Install Nashpy

Nashpy currently requires Python 3.5 or above. To install from the Python Package index (PyPi) run the following command:

```
$ pip install nashpy
```

To install a development version from source:

```
$ git clone https://github.com/drvinceknight/Nashpy.git
$ cd nashpy
$ python setup.py develop
```

## 2.2 Create a game

A game in Nashpy is created by passing 1 or 2 matrices to the `nash.Game` class. Here is the zero sum game matching pennies:

```
>>> import nashpy as nash
>>> import numpy as np
>>> A = np.array([[1, -1], [-1, 1]])
>>> matching_pennies = nash.Game(A)
>>> matching_pennies
Zero sum game with payoff matrices:

Row player:
[[ 1 -1]
```

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```

[-1  1]]

Column player:
[[-1  1]
 [ 1 -1]]

```

Here is the **non** zero sum game prisoners dilemma:

```

>>> import nashpy as nash
>>> import numpy as np
>>> A = np.array([[3, 0], [5, 1]])
>>> B = np.array([[3, 5], [0, 1]])
>>> prisoners_dilemma = nash.Game(A, B)
>>> prisoners_dilemma
Bi matrix game with payoff matrices:

Row player:
[[3 0]
 [5 1]]

Column player:
[[3 5]
 [0 1]]

```

## 2.3 Calculate utilities

A game can be passed a pair of mixed strategies (distributions over the set of pure strategies) to return the utilities. Let us create a game to illustrate this:

```

>>> import nashpy as nash
>>> import numpy as np
>>> A = np.array([[3, 0], [5, 1]])
>>> B = np.array([[3, 5], [0, 1]])
>>> prisoners_dilemma = nash.Game(A, B)

```

The utility for both players when they both play their first strategy:

```

>>> sigma_r = np.array([1, 0])
>>> sigma_c = np.array([1, 0])
>>> prisoners_dilemma[sigma_r, sigma_c]
array([3, 3])

```

The utility to both players when they play uniformly randomly across both their strategies:

```

>>> sigma_r = np.array([1 / 2, 1 / 2])
>>> sigma_c = np.array([1 / 2, 1 / 2])
>>> prisoners_dilemma[sigma_r, sigma_c]
array([ 2.25,  2.25])

```

## 2.4 Solve with support enumeration

One of the algorithms implemented in Nashpy is called *Support enumeration*, this is implemented as a method on the Game class:

```
>>> import nashpy as nash
>>> import numpy as np
>>> A = np.array([[1, -1], [-1, 1]])
>>> matching_pennies = nash.Game(A)
```

This `support_enumeration` method returns a generator of all the equilibria:

```
>>> equilibria = matching_pennies.support_enumeration()
>>> for eq in equilibria:
...     print(eq)
(array([ 0.5,  0.5]), array([ 0.5,  0.5]))
```

## 2.5 Solve with vertex enumeration

One of the algorithms implemented in Nashpy is called *Vertex enumeration*, this is implemented as a method on the Game class:

```
>>> import nashpy as nash
>>> import numpy as np
>>> A = np.array([[1, -1], [-1, 1]])
>>> matching_pennies = nash.Game(A)
```

This `vertex_enumeration` method returns a generator of all the equilibria:

```
>>> equilibria = matching_pennies.vertex_enumeration()
>>> for eq in equilibria:
...     print(eq)
(array([ 0.5,  0.5]), array([ 0.5,  0.5]))
```

## 2.6 Solve with Lemke Howson

One of the algorithms implemented in Nashpy is *The Lemke Howson Algorithm*. This algorithm does not return **all** equilibria and takes an input argument:

```
>>> import nashpy as nash
>>> import numpy as np
>>> A = np.array([[1, -1], [-1, 1]])
>>> matching_pennies = nash.Game(A)
>>> matching_pennies.lemke_howson(initial_dropped_label=0)
(array([ 0.5,  0.5]), array([ 0.5,  0.5]))
```

The `initial_dropped_label` is an integer between 0 and  $\text{sum}(A.\text{shape}) - 1$ . To iterate over all possible labels use the `lemke_howson_enumeration` which returns a generator:

```
>>> equilibria = matching_pennies.lemke_howson_enumeration()
>>> for eq in equilibria:
```

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```
...     print(eq)
(array([ 0.5,  0.5]), array([ 0.5,  0.5]))
(array([ 0.5,  0.5]), array([ 0.5,  0.5]))
(array([ 0.5,  0.5]), array([ 0.5,  0.5]))
(array([ 0.5,  0.5]), array([ 0.5,  0.5]))
```

Note that this algorithm is not guaranteed to find **all** equilibria but is an efficient way of finding **an** equilibrium.

### 3.1 Support enumeration

The support enumeration algorithm implemented in `Nashpy` is based on the one described in [Nisan2007].

The algorithm is as follows:

For a degenerate 2 player game  $(A, B) \in \mathbb{R}^{m \times n^2}$  the following algorithm returns all nash equilibria:

1. For all  $1 \leq k_1 \leq m$  and  $1 \leq k_2 \leq n$ ;
2. For all pairs of support  $(I, J)$  with  $|I| = k_1$  and  $|J| = k_2$ .
3. Solve the following equations (this ensures we have best responses):

$$\sum_{i \in I} \sigma_{r_i} B_{ij} = v \text{ for all } j \in J$$

$$\sum_{j \in J} A_{ij} \sigma_{c_j} = u \text{ for all } i \in I$$

4. Solve
  - $\sum_{i=1}^m \sigma_{r_i} = 1$  and  $\sigma_{r_i} \geq 0$  for all  $i$
  - $\sum_{j=1}^n \sigma_{c_j} = 1$  and  $\sigma_{c_j} \geq 0$  for all  $j$
5. Check the best response condition.

Repeat steps 3,4 and 5 for all potential support pairs.

#### 3.1.1 Discussion

1. Step 1 is a complete enumeration of all possible strategies that the equilibria could be.
2. Step 2 can be modified to only consider degenerate games ensuring that only supports of equal size are considered  $|I| = |J|$ . This is described further in *Degenerate games*.

3. Step 3 are the linear equations that are to be solved, for a given pair of supports these ensure that neither player has an incentive to move to another strategy on that support.
4. Step 4 is to ensure we have mixed strategies.
5. Step 5 is a final check that there is no better utility outside of the supports.

In `Nashpy` this is all implemented algebraically using `Numpy` to solve the linear equations.

## 3.2 Vertex enumeration

The vertex enumeration algorithm implemented in `Nashpy` is based on the one described in [Nisan2007].

The algorithm is as follows:

For a nondegenerate 2 player game  $(A, B) \in \mathbb{R}^{m \times n^2}$  the following algorithm returns all nash equilibria:

1. Obtain the best response Polytopes  $P$  and  $Q$ .
2. For all pairs of vertices of  $P$  and  $Q$ .
3. Check if the pair is fully labeled and return the normalised probability vectors.

Repeat steps 2 and 3 for all pairs of vertices.

### 3.2.1 Discussion

1. Before creating the best response Polytope we need to consider the best response Polyhedron. For the row player, this corresponds to the set of all the mixed strategies available to the row player as well as an upper bound on the utilities to the column player. Analogously for the column player:

$$\begin{aligned}\bar{P} &= \{(x, v) \in \mathbb{R}^m \times \mathbb{R} \mid x \geq 0, \mathbb{1}^T x = 1, B^T x \leq \mathbb{1}^T v\} \\ \bar{Q} &= \{(y, u) \in \mathbb{R}^n \times \mathbb{R} \mid y \geq 0, \mathbb{1}^T y = 1, Ay \leq \mathbb{1}^T u\}\end{aligned}$$

Note that in both definitions above we have a total of  $m + n$  inequalities in the constraints.

For  $P$ , the first  $m$  of those constraints correspond to the elements of  $x$  being greater or equal to 0. For a given  $x$ , if  $x_i = 0$ , we say that  $x$  has label `i`. This corresponds to strategy  $i$  not being in the support of  $x$ .

For the last  $n$  of these inequalities, when they are equalities they correspond to whether or not strategy  $1 \leq j \leq n$  of the other player is a best response to  $x$ . Similarly, if strategy  $j$  is a best response to  $x$  then we say that  $x$  has label  $m + j$ .

This all holds analogously for the column player. If the labels of a pair of elements of  $\bar{P}$  and  $\bar{Q}$  give the full set of integers from 1 to  $m + n$  then they represent strategies that are best responses to each other. Since, this would imply that either a pure strategy is not played or it is a best response to the other players strategy.

The difficulty with using the best response Polyhedron is that the upper bound on the utilities of both players  $(u, v)$  is not known. Importantly, we do not need to know it. Thus, we assume that in both cases:  $u = v = 1$  (this simply corresponds to a scaling of our strategy vectors).

This allows us to define the best response Polytopes:

$$\begin{aligned}P &= \{(x, v) \in \mathbb{R}^m \times \mathbb{R} \mid x \geq 0, B^T x \leq 1\} \\ Q &= \{(y, u) \in \mathbb{R}^n \times \mathbb{R} \mid y \geq 0, Ay \leq 1\}\end{aligned}$$

2. Step 2: The vertices of these polytopes are the points that will have labels (they are the points that lie at the intersection of the underlying halfspaces [Ziegler2012]).



To find these vertices, `nashpy` uses `scipy` which has a handy class for creating Polytopes using the inequality definitions and being able to return the vertices. Here is the wrapper written in `nashpy` that is used by the vertex enumeration algorithm to give the vertices and corresponding labels:

```
>>> import nashpy as nash
>>> import numpy as np
>>> A = np.array([[3, 1], [1, 3]])
>>> halfspaces = nash.polytope.build_halfspaces(A)
>>> vertices = nash.polytope.non_trivial_vertices(halfspaces)
>>> for vertex in vertices:
...     print(vertex)
(array([ 0.333...,  0...]), {0, 3})
(array([ 0...,  0.333...]), {1, 2})
(array([ 0.25,  0.25]), {0, 1})
```

- Step 3, we iterate over all pairs of the vertices of both polytopes and pick out the ones that are fully labeled. Because of the scaling that took place to create the Polytope from the Polyhedron, we will need to return a normalisation of both vertices.

### 3.3 The Lemke Howson Algorithm

The Lemke Howson algorithm implemented in `Nashpy` is based on the one described in [Nisan2007] originally introduced in [Lemke1964].

The algorithm is as follows:

For a nondegenerate 2 player game  $(A, B) \in \mathbb{R}^{m \times n^2}$  the following algorithm returns a single Nash equilibria:

- Obtain the best response Polytopes  $P$  and  $Q$ .
- Choose a starting label to drop, this will correspond to a vertex of  $P$  or  $Q$ .
- In that polytope, remove the label from the corresponding vertex and move to the vertex that shared that label. A new label will be picked up and duplicated in the other polytope.
- In the other polytope drop the duplicate label and move to the vertex that shared that label.

Repeat steps 3 and 4 until there are no duplicate labels.

#### 3.3.1 Discussion

This algorithm is implemented using integer pivoting.

- Step 1, the best response polytopes  $P$  and  $Q$  are represented by a tableau. For example for:

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$$

This is represented as a pair of tableau:

$$T_c = \begin{pmatrix} 3 & 1 & 1 & 0 & 1 \\ 1 & 3 & 0 & 1 & 1 \end{pmatrix}$$

For reasons that will become clear, we in fact shift this tableau so that the labelling is coherent across both polytopes:

$$T_c = \begin{pmatrix} 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & 1 & 3 & 1 \end{pmatrix}$$

Here it is as a numpy array:

```
>>> import numpy as np
>>> col_tableau = np.array([[1, 0, 3, 1, 1],
...                        [0, 1, 1, 3, 1]])
```

Here is the tableau that corresponds to  $B$ :

$$T_r = \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 3 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Here it is as a numpy array:

```
>>> row_tableau = np.array([[1, 2, 1, 0, 1],
...                        [3, 1, 0, 1, 1]])
```

- Step 2, choosing a starting label is choosing an integer from  $0 \leq k < m + n$  (we start our indices at 0). As an example, let us choose the label 1.

First we need to identify which vertex has that label. The labels of a tableau correspond to the non basic variables: these are the columns with more than just a single non zero variable:

- The labels of  $T_c$  are thus  $\{2, 3\}$ :

```
>>> import nashpy as nash
>>> nash.integer_pivoting.non_basic_variables(col_tableau)
{2, 3}
```

- The labels of  $T_r$  are thus  $\{0, 1\}$ :

```
>>> nash.integer_pivoting.non_basic_variables(row_tableau)
{0, 1}
```

So we are going to drop label 1 from  $T_r$ .

- Step 3, removing a label and moving from one vertex to another corresponds to integer pivoting [Dantzig2016]. This is a manipulation of  $T$ , dropping label 1 corresponds to pivoting the second column.

To do this we need to identify which row will not change (the “pivot row”), this is done by finding the smallest ratio of value in that column over the value in the last column:  $(T_r)_{i4}/(T_r)_{ik}$ .

In our case the first row has corresponding ratio:  $1/2$  and the second ratio  $1/1$ . So our pivot row is the first row:

```
>>> nash.integer_pivoting.find_pivot_row(row_tableau, column_index=1)
0
```

What we now do is row operations so as to make the second column correspond to a basic variable. We will do this by multiplying the second row by 2 and then subtracting the first row by it:

$$T_r = \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 5 & 0 & -1 & 2 & 1 \end{pmatrix}$$

Our resulting tableau has labels:  $\{0, 2\}$  so it has “picked up” label 2:

```
>>> nash.integer_pivoting.pivot_tableau(row_tableau, column_index=1)
{2}
>>> row_tableau
array([[ 1,  2,  1,  0,  1],
       [ 5,  0, -1,  2,  1]])
```

4. Step 4, we will now repeat the previous manipulation on  $T_c$  where we now need to drop the duplicate label 2. We do this by pivoting the third column.

The ratios are:  $1/3$  for the first row and  $1/1$  for the second, thus the pivot row is the first row:

```
>>> nash.integer_pivoting.find_pivot_row(col_tableau, column_index=2)
0
```

Using similar row operations we obtain:

$$T_c = \begin{pmatrix} 1 & 0 & 3 & 1 & 1 \\ -1 & 3 & 0 & 8 & 2 \end{pmatrix}$$

Our resulting tableau has labels:  $\{0, 3\}$ , so it has picked up label 0:

```
>>> nash.integer_pivoting.pivot_tableau(col_tableau, column_index=2)
{0}
>>> col_tableau
array([[ 1,  0,  3,  1,  1],
       [-1,  3,  0,  8,  2]])
```

We now need to drop 0 from  $T_r$ , we do this by pivoting the first column. The ratio test:  $1/1 > 1/5$  implies that the second row is the pivot row. Using similar algebraic manipulations we obtain:

$$T_r = \begin{pmatrix} 0 & 10 & 6 & -2 & 4 \\ 5 & 0 & -1 & 2 & 1 \end{pmatrix}$$

Our resulting tableau has labels:  $\{2, 3\}$ , so it has picked up label 3:

```
>>> nash.integer_pivoting.pivot_tableau(row_tableau, column_index=0)
{3}
>>> row_tableau
array([[ 0, 10,  6, -2,  4],
       [ 5,  0, -1,  2,  1]])
```

We now need to drop 3 from  $T_c$ , we do this by pivoting the fourth column. The ratio test:  $1/1 > 2/8$  indicates that we pivot on the second row which gives:

$$T_c = \begin{pmatrix} 9 & -1 & 24 & 0 & 6 \\ -1 & 3 & 0 & 8 & 2 \end{pmatrix}$$

Our resulting tableau has labels:  $\{0, 1\}$ :

```
>>> nash.integer_pivoting.pivot_tableau(col_tableau, column_index=3)
{1}
>>> col_tableau
array([[ 9, -3, 24,  0,  6],
       [-1,  3,  0,  8,  2]])
```

The union of the labels of  $T_r$  and  $T_c$  is:  $\{0, 1, 2, 3\}$  which implies that we have a fully labeled vertex pair.

The vertex corresponding to  $T_r$  are obtained by setting the non basic variables to 0 and looking at the corresponding values of the first two columns:

$$v_r = (1/5, 4/10) = (1/5, 2/5)$$

The vertex corresponding to  $T_c$  are obtained from the last 2 columns:

$$v_c = (6/24, 2/8) = (1/4, 1/4)$$

The final step of the algorithm is to return the normalised probabilities that correspond to these vertices:

$$((1/3, 2/3), (1/2, 1/2))$$

### 3.4 Degenerate games

A two player game is called nondegenerate if no mixed strategy of support size  $k$  has more than  $k$  pure best responses.

For example, the zero sum game defined by the following matrix is degenerate:

$$A = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

The third column has two pure best responses.

When dealing with *degenerate* games unexpected results can occur:

```
>>> import nashpy as nash
>>> import numpy as np
>>> A = np.array([[0, -1, 1], [-1, 0, 1], [-1, 0, 1]])
>>> game = nash.Game(A)
```

Here is the output when using *Support enumeration*:

```
>>> for eq in game.support_enumeration():
...     print(np.round(eq[0], 2), np.round(eq[1], 2))
[ 0.5  0.5  0. ] [ 0.5  0.5  0. ]
[ 0.5  0.   0.5] [ 0.5  0.5  0. ]
```

Here is the output when using *Vertex enumeration*:

```
>>> for eq in game.vertex_enumeration():
...     print(np.round(eq[0], 2), np.round(eq[1], 2))
[ 0.5  0.   0.5] [ 0.5  0.5 -0. ]
[ 0.5  0.5 -0. ] [ 0.5  0.5 -0. ]
```

Here is the output when using the *The Lemke Howson Algorithm*:

```
>>> for eq in game.lemke_howson_enumeration():
...     print(np.round(eq[0], 2), np.round(eq[1], 2))
[ 0.33...  0.33...  0.33...] [ nan]
```

We see that the *lemke-howson* algorithm fails but also that the *Support enumeration* and *Vertex enumeration* fail to find some equilibria: there is in fact a range of strategies the row player can play against  $[0.5 \ 0.5 \ 0]$  that is still a best response.

The *Support enumeration* algorithm can be executed with two optional arguments that allow for control of its execution:

- `non_degenerate=True` (False is the default) will only consider supports of equal size. If you know your game is non degenerate this will make support enumeration execute less checks.

- `tol=0` (`10 ** -16` is the default), when considering the underlying linear system `tol` is considered to be a lower bound for difference between two real numbers. Using `tol=0` ensures a very strict execution of the algorithm.

Here is an example:

```
>>> A = np.array([[4, 9, 9], [9, 1, 6], [9, 2, 3]])
>>> B = np.array([[2, 2, 5], [7, 4, 4], [1, 6, 4]])
>>> game = nash.Game(A, B)
>>> for eq in game.support_enumeration():
...     print(np.round(eq[0], 2), np.round(eq[1], 2))
[ 1.  0.  0.] [ 0.  0.  1.]
[ 0.  1.  0.] [ 1.  0.  0.]
[ 0.5  0.5  0. ] [ 0.38  0.   0.62]
[ 0.2  0.5  0.3] [ 0.57  0.32  0.11]
>>> for eq in game.support_enumeration(non_degenerate=True):
...     print(np.round(eq[0], 2), np.round(eq[1], 2))
[ 1.  0.  0.] [ 0.  0.  1.]
[ 0.  1.  0.] [ 1.  0.  0.]
[ 0.2  0.5  0.3] [ 0.57  0.32  0.11]
>>> for eq in game.support_enumeration(non_degenerate=False, tol=0):
...     print(np.round(eq[0], 2), np.round(eq[1], 2))
[ 1.  0.  0.] [ 0.  0.  1.]
[ 0.  1.  0.] [ 1.  0.  0.]
[ 0.2  0.5  0.3] [ 0.57  0.32  0.11]
```

## 3.5 Bibliography

This is a collection of various bibliographic items referenced in the documentation.

## 3.6 Source files

### 3.6.1 Subpackages

**nash.algorithms package**

**Submodules**

**nashpy.algorithms.support\_enumeration module**

A class for a normal form game

`nashpy.algorithms.support_enumeration.indifference_strategies` (*A*, *B*,  
*non\_degenerate=False*,  
*tol=1e-16*)

A generator for the strategies corresponding to the potential supports

#### Returns

- A generator of all potential strategies that are indifferent on each
- potential support. Return *False* if they are not valid (not a
- probability vector OR not fully on the given support).

`nashpy.algorithms.support_enumeration.is_ne` (*strategy\_pair*, *support\_pair*, *payoff\_matrices*)

Test if a given strategy pair is a pair of best responses

**Parameters**

- **strategy\_pair** (a 2-tuple of numpy arrays)–
- **support\_pair** (a 2-tuple of numpy arrays)–

`nashpy.algorithms.support_enumeration.obey_support` (*strategy*, *support*, *tol=1e-16*)

Test if a strategy obeys its support

**Parameters**

- **strategy** (a numpy array)– A given strategy vector
- **support** (a numpy array)– A strategy support

**Returns**

- **A boolean** (whether or not that strategy does indeed have the given support)

`nashpy.algorithms.support_enumeration.potential_support_pairs` (*A*, *B*, *non\_degenerate=False*)

A generator for the potential support pairs

**Returns**

**Return type** A generator of all potential support pairs

`nashpy.algorithms.support_enumeration.powerset` (*n*)

A power set of range(n)

Based on recipe from python itertools documentation:

<https://docs.python.org/2/library/itertools.html#recipes>

`nashpy.algorithms.support_enumeration.solve_indifference` (*A*, *rows=None*, *columns=None*)

Solve the indifference for a payoff matrix assuming support for the strategies given by columns

Finds vector of probabilities that makes player indifferent between rows. (So finds probability vector for corresponding column player)

**Parameters**

- **A** (a 2 dimensional numpy array (A payoff matrix for the row player))–
- **rows** (the support played by the row player)–
- **columns** (the support player by the column player)–

**Returns**

- A numpy array
- A probability vector for the column player that makes the row player indifferent. Will return False if all entries are not >= 0.

`nashpy.algorithms.support_enumeration.support_enumeration` (*A*, *B*, *non\_degenerate=False*, *tol=1e-16*)

Obtain the Nash equilibria using support enumeration.

Algorithm implemented here is Algorithm 3.4 of [Nisan2007]

1. For each  $k$  in  $1 \dots \min(\text{size of strategy sets})$
2. For each  $I, J$  supports of size  $k$
3. Solve indifference conditions
4. Check that have Nash Equilibrium.

**Returns equilibria**

**Return type** A generator.

### nashpy.algorithms.vertex\_enumeration module

A class for the vertex enumeration algorithm

`nashpy.algorithms.vertex_enumeration.vertex_enumeration(A, B)`

Obtain the Nash equilibria using enumeration of the vertices of the best response polytopes.

Algorithm implemented here is Algorithm 3.5 of [Nisan2007]

1. Build best responses polytopes of both players
2. For each vertex pair of both polytopes
3. Check if pair is fully labelled
4. Return the normalised pair

**Returns equilibria**

**Return type** A generator.

### nashpy.algorithms.lemke\_howson module

A class for the Lemke Howson algorithm

`nashpy.algorithms.lemke_howson.lemke_howson(A, B, initial_dropped_label=0)`

Obtain the Nash equilibria using the Lemke Howson algorithm implemented using integer pivoting.

Algorithm implemented here is Algorithm 3.6 of [Nisan2007].

1. Start at the artificial equilibrium (which is fully labeled)
2. Choose an initial label to drop and move in the polytope for which the vertex has that label to the edge that does not share that label. (This is implemented using integer pivoting)
3. A label will now be duplicated in the other polytope, drop it in a similar way.
4. Repeat steps 2 and 3 until have Nash Equilibrium.

**Parameters** `initial_dropped_label` (*int*) –

**Returns equilibria**

**Return type** A tuple.

`nashpy.algorithms.lemke_howson.shift_tableau(tableau, shape)`

Shift a tableau to ensure labels of pairs of tableaux coincide

**Parameters**

- **tableau** (*a numpy array*) –
- **shape** (*a tuple*) –

**Returns tableau**

**Return type** a numpy array

`nashpy.algorithms.lemke_howson.tableau_to_strategy` (*tableau, basic\_labels, strategy\_labels*)

Return a strategy vector from a tableau

**Parameters**

- **tableau** (*a numpy array*) –
- **basic\_labels** (*a set*) –
- **strategy\_labels** (*a set*) –

**Returns strategy**

**Return type** a numpy array

### 3.6.2 Submodules

### 3.6.3 nashpy.game module

A class for a normal form game

**class** `nashpy.game.Game` (\*args)

Bases: `object`

A class for a normal form game.

**Parameters**

- **A, B** (-) – non zero sum games.
- **A** (-) – zero sum game.

**lemke\_howson** (*initial\_dropped\_label*)

Obtain the Nash equilibria using the Lemke Howson algorithm implemented using integer pivoting.

Algorithm implemented here is Algorithm 3.6 of [Nisan2007].

1. Start at the artificial equilibrium (which is fully labeled)
2. Choose an initial label to drop and move in the polytope for which the vertex has that label to the edge that does not share that label. (This is implemented using integer pivoting)
3. A label will now be duplicated in the other polytope, drop it in a similar way.
4. Repeat steps 2 and 3 until have Nash Equilibrium.

**Parameters** **initial\_dropped\_label** (*int*) –

**Returns** **equilibria**

**Return type** A tuple.



**lemke\_howson\_enumeration()**

Obtain Nash equilibria for all possible starting dropped labels using the lemke howson algorithm. See *Game.lemke\_howson* for more information.

Note: this is not guaranteed to find all equilibria.

**Returns equilibria**

**Return type** A generator

**support\_enumeration** (*non\_degenerate=False, tol=1e-16*)

Obtain the Nash equilibria using support enumeration.

Algorithm implemented here is Algorithm 3.4 of [Nisan2007].

1. For each  $k$  in  $1 \dots \min(\text{size of strategy sets})$
2. For each I,J supports of size  $k$
3. Solve indifference conditions
4. Check that have Nash Equilibrium.

**Returns equilibria**

**Return type** A generator.

**vertex\_enumeration()**

Obtain the Nash equilibria using enumeration of the vertices of the best response polytopes.

Algorithm implemented here is Algorithm 3.5 of [Nisan2007].

1. Build best responses polytopes of both players
2. For each vertex pair of both polytopes
3. Check if pair is fully labelled
4. Return the normalised pair

**Returns equilibria**

**Return type** A generator.

### 3.6.4 Module contents



## 4.1 John Nash

This library is named after the mathematician John Nash. He is most famous for his work in Game Theory that culminated in him winning a Noble prize in Economics. The book [Nasar2011] (popularized in a 2001 movie) gives a good overview of his life.

The work he received a Noble prize for was a proof that a game **always** has an equilibrium [Nash1950]. His proof is an exceptional piece of mathematics where he uses a fixed point theorem by showing that an equilibrium is equivalent to a fixed point of a function.

Subsequently, these equilibria have been referred to as Nash equilibria.

## 4.2 How does Nashpy relate to Gambit

*Gambit* is the state of the art software library for Game Theory [McKelvey2016]. It also has a Python interface. It handles  $N \geq 2$  player games and is computationally efficient. It is a much more mature piece of software than *Nashpy*.

It does **however** sometimes prove difficult to install (because of the required C libraries), in particular installation is not supported on Windows. In those instances you can use *Game Theory Explorer* which is a great web point and click Graphical User Interface (GUI) to *Gambit*.

The main mission statement of *Nashpy* is to provide a simple to install Python library that implements algorithms that are simple to implement using the scientific Python stack (*numpy* and *scipy*).

This is motivated by the fact that I wanted a Python library (not a GUI as I am keen to teach reproducibly research methodologies) for teaching my Mathematics students. Using the *Gambit* Python interface is not sufficient for this as students need to be able to install it on their own machines (without difficulty).

All the algorithms in *Nashpy* are implemented with readability as the main motivation. This at times comes at an efficiency cost. For example, *Support enumeration* builds the entire Polytope representation (using functionality of *scipy*) which is not efficient.

**To summarise:**

- If you want to do sophisticated efficient game theoretic computations, use [Gambit](#).
- If you are happy to use a GUI use [Game Theory Explorer](#).
- If you would like an easy to install Python library for two player games you can use `Nashpy`.

### 4.3 Other Python Game theory libraries

- [Axelrod](#): a research library aimed at the study of the Iterated Prisoners dilemma [[Knight2016](#)].
- [Gambit](#): a C library with a Python interface for the computation of equilibria [[McKelvey2016](#)]. *How does Nashpy relate to Gambit.*
- [Game theory explorer](#) a web interface to gambit useful for teaching. [[Savani2015](#)]
- [PyNFG](#): PyNFG is a Python package for modeling and solving Network Form Games.
- [lrslib](#): A C implementation of a reverse search algorithm with modules for Nash equilibria computation.
- [sagemath](#): The mathematical software package Sage has various algorithms for the computation of Nash equilibria.

## CHAPTER 5

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### Indices and tables

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- `genindex`
- `modindex`
- `search`



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